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Asymptotic Properties of Zeros of Hypergeometric Polynomials[1]

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In a paper by K. Driver and P. Duren (1999, *Numer. Algorithms* **21**, 147–156) a theorem of Borwein and Chen was used to show that for each $k \in \mathbb{N}$ the zeros of the hypergeometric polynomials $F(-n, kn+1; kn+2; z)$ cluster on the loop of the lemniscate $\{z : |z^k(1-z)| = k^k/(k+1)^{k+1}\}$, with $Re\{z\} > k/(k+1)$ as $n \to \infty$. We now supply a direct proof which generalizes this result to arbitrary $k > 0$, while showing that every point of the curve is a cluster point of zeros. Examples generated by computer graphics suggest some finer asymptotic properties of the zeros. © 2001 Academic Press

Key Words: hypergeometric polynomials; zeros; asymptotics; lemniscates.

1. INTRODUCTION

The Gauss hypergeometric function is defined as

$$F(a, b; c; z) = 1 + \sum_{m=1}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \qquad |z| < 1,$$

where

$$(\alpha)_m = \alpha(\alpha+1) \cdots (\alpha+m-1) = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}$$

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is Pochhammer's symbol. If $a = -n$ is a negative integer, the series terminates and reduces to a polynomial of degree n, called a hypergeometric polynomial. Hypergeometric functions have the Euler integral representation

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\,(1-zt)^{-a}\,dt$$

for $Re\{c\} > Re\{b\} > 0$.

Choosing $a = -n$ and $b = c - 1 = kn + 1$ for $n \in \mathbb{N}$ and arbitrary $k > 0$, we arrive at the representation

$$F(-n, kn+1; kn+2; z) = (kn+1) \int_0^1 [t^k(1-zt)]^n\,dt$$

for the class of hypergeometric polynomials to be treated in this paper. For fixed $z \in \mathbb{C}$, we regard

$$f(t) = t^k(1-zt)$$

as a function of the complex variable t. Note that $f(t)$ has zeros at 0 and $\frac{1}{z}$, while $f'(t)$ has a zero at $\frac{k}{(k+1)z}$. Thus the surface $|f(t)|$ has a saddle point at $\frac{k}{(k+1)z}$.

The main purpose of this paper is to prove the following theorem.

THEOREM. *For each real number $k > 0$, the zeros of the hypergeometric polynomials $F(-n, kn+1; kn+2; z)$ approach the half-lemniscate*

$$\left\{ z : |z^k(1-z)| = \frac{k^k}{(k+1)^{k+1}},\ Re\{z\} > \frac{k}{k+1} \right\}$$

as $n \to \infty$. Furthermore, every point of this curve is a cluster point of zeros.

In the special case where k is a positive integer, Driver and Duren [4] invoked a theorem of Borwein and Chen [1] to show that the zeros cluster on the right-hand loop of the lemniscate. On the basis of graphical evidence, they conjectured the generalization to arbitrary $k > 0$, where the argument of Borwein and Chen does not apply. Our contribution in the present paper is to give a complete proof, more explicit than that of Borwein and Chen, which applies to arbitrary $k > 0$ and gives further information.

2. THE SADDLE-POINT METHOD

In order to discuss the asymptotic behavior of the zeros of the hypergeometric polynomials $F(-n, kn+1; kn+2; z)$, we shall use the saddle-point method (cf. de Bruijn [2] or Copson [3]) to get an asymptotic expansion for the integral

$$\int_0^1 [f(t)]^n \, dt, \qquad f(t) = t^k(1-zt).$$

We begin with the following lemma.

LEMMA 1. (a) *The inequality* $|f(1)| > |f(\frac{k}{(k+1)z})|$ *holds if and only if* $|z^k(1-z)| > k^k/(k+1)^{k+1}$. *Similarly,* $|f(1)| < |f(\frac{k}{(k+1)z})|$ *if and only if* $|z^k(1-z)| < k^k/(k+1)^{k+1}$.

(b) *If* $Re\{z\} > \frac{k}{k+1}$, *the function* $|f(t)|$ *has a unique path of steepest ascent from* $\frac{1}{z}$ *to* 1. *If* $Re\{z\} < \frac{k}{k+1}$, *there is a unique path of steepest ascent from* 0 *to* 1.

Proof. (a) The condition $|f(1)| > |f(\frac{k}{(k+1)z})|$ says that $|1-z| > |(\frac{k}{(k+1)z})^k(1-\frac{k}{k+1})|$, which reduces to $|z^k(1-z)| > k^k/(k+1)^{k+1}$. Geometrically, the statement is that $|f(1)| > |f(\frac{k}{(k+1)z})|$ for z outside the lemniscate, while the inequality is reversed when z is inside either loop of the lemniscate.

(b) The line through the saddle-point $\frac{k}{(k+1)z}$ perpendicular to the linear segment from 0 to $\frac{1}{z}$ is a kind of "continental divide" that separates the plane into two basins containing 0 and $\frac{1}{z}$, respectively. Any point in the 0-basin is joined to 0 by a unique path of steepest descent, orthogonal to the level curves of $f(t)$; points in the $\frac{1}{z}$-basin can be similarly joined to $\frac{1}{z}$. (See Fig. 1.) To prove (b), we have to show that the point 1 lies in the $\frac{1}{z}$-basin if and only if $Re\{z\} > \frac{k}{k+1}$. For this purpose we multiply by $\frac{z}{|z|}$, which rotates the figure so that both zeros move to the real axis and the line through the saddle-point is carried to the vertical line through the point $\frac{k}{(k+1)|z|}$. Thus the point 1 is in the $\frac{1}{z}$-basin if and only if

$$Re\left\{\frac{z}{|z|}\right\} > \frac{k}{(k+1)|z|},$$

which is equivalent to the condition $Re\{z\} > \frac{k}{k+1}$. The same analysis shows that the point 1 is in the 0-basin if and only if $Re\{z\} < \frac{k}{k+1}$. This proves the lemma.

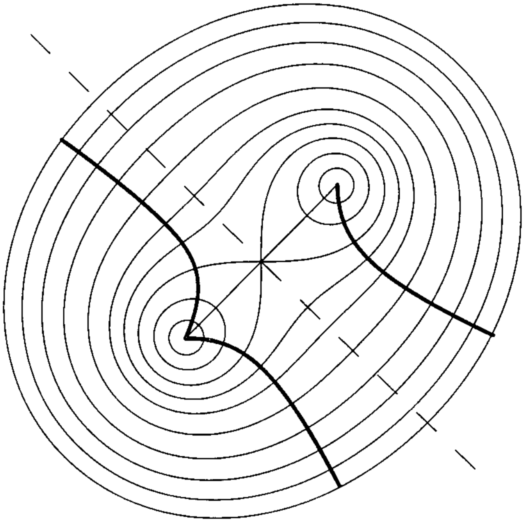

FIG. 1. Points in the 0-basin and in the $\frac{1}{z}$-basin can be joined by orthogonal trajectories only to 0 and $\frac{1}{z}$, respectively.

If $Re\{z\} > \frac{k}{k+1}$, we deform the path of integration to write

$$\int_0^1 [f(t)]^n \, dt = \int_0^{1/z} [f(t)]^n \, dt + \int_{1/z}^1 [f(t)]^n \, dt,$$

following the linear path from 0 to $\frac{1}{z}$ and the path of steepest ascent (guaranteed by Lemma 1) from $\frac{1}{z}$ to 1. The linear path from 0 to $\frac{1}{z}$ is orthogonal to the level curves of $f(t)$ and is therefore the path of steepest ascent from 0 to the saddle-point $\frac{k}{(k+1)z}$, followed by the path of steepest descent to $\frac{1}{z}$. To compute the first integral we make the substitution $t = \frac{s}{z}$ for $0 \leqslant s \leqslant 1$, so that

$$\int_0^{1/z} [f(t)]^n \, dt = \frac{1}{z^{kn+1}} \int_0^1 [s^k(1-s)]^n \, ds = \frac{n!}{z^{kn+1}} \frac{\Gamma(kn+1)}{\Gamma((k+1)n+2)}. \quad (1)$$

The second integral is more difficult to handle. We first treat the special case $k=1$, where a complete asymptotic series can be calculated explicitly. We will therefore assume first that $Re\{z\} > \frac{1}{2}$. In order to find the path of steepest ascent from $\frac{1}{z}$ to 1, we use the fact that $f(t)$ will have constant argument along this path (see Copson [3, p. 65]). Thus we can parameterize the path by letting $f(t) = f(1)\, r$, or

$$t(1-zt) = r(1-z), \qquad 0 \leqslant r \leqslant 1.$$

Solving this quadratic equation yields

$$t = \frac{1 + \sqrt{1 - 4z(1-z)\,r}}{2z},$$

where the branch of the square root is chosen for which $\sqrt{1} = 1$ in order to make $r = 0$ correspond to $t = \frac{1}{z}$. Note, however, that $t = 1$ when $r = 1$, so we must interpret

$$\sqrt{1 - 4z(1-z)} = \sqrt{(1-2z)^2} = -(1-2z).$$

To transform the integral, we observe that $(1 - 2zt)\,dt = (1-z)\,dr$, or

$$dt = -\frac{1-z}{\sqrt{1 - 4z(1-z)\,r}}\,dr,$$

choosing the branch of the square root as indicated above. Then

$$\begin{aligned}\int_{1/z}^{1} [f(t)]^n\,dt &= -\int_0^1 [r(1-z)]^n \frac{1-z}{\sqrt{1-4z(1-z)\,r}}\,dr \\ &= -(1-z)^{n+1} \int_0^1 \frac{r^n}{\sqrt{1-4z(1-z)\,r}}\,dr. \qquad (2)\end{aligned}$$

If $Re\{z\} < \frac{k}{k+1}$, we again consider first the special case $k = 1$, so that $Re\{z\} < \frac{1}{2}$. Then we may evaluate the integral $\int_0^1 [f(t)]^n\,dt$ over the path of steepest ascent from 0 to 1. Again let $f(t) = f(1)\,r$, $0 < r < 1$. Now, however, it is convenient to write

$$t = \frac{1 - \sqrt{1 - 4z(1-z)\,r}}{2z},$$

so that the branch of the square root with $\sqrt{1} = 1$ will give $t = 0$ when $r = 0$. To make $t = 1$ when $r = 1$, we must now interpret

$$\sqrt{1 - 4z(1-z)} = 1 - 2z.$$

The result is

$$\int_0^1 [f(t)]^n\,dt = (1-z)^{n+1} \int_0^1 \frac{r^n}{\sqrt{1-4z(1-z)\,r}}\,dr. \qquad (3)$$

For general $k>0$ the calculations of the integrals (2) and (3) are very similar. For $Re\{z\} > \frac{k}{k+1}$ we write

$$f(t)=t^k(1-zt)=(1-z)\,r, \qquad 0\leqslant r\leqslant 1, \tag{4}$$

so that

$$t^{k-1}[k-(k+1)\,zt]\,dt=(1-z)\,dr$$

and

$$\int_{1/z}^{1} [f(t)]^n\,dt=(1-z)^{n+1}\int_0^1 \frac{r^n}{t^{k-1}[k-(k+1)\,zt]}\,dr, \tag{5}$$

where $t=t(r)$ is a function determined implicitly by (4), with $t(0)=\frac{1}{z}$ and $t(1)=1$. For $Re\{z\} < \frac{k}{k+1}$, we find

$$\int_0^1 [f(t)]^n\,dt=(1-z)^{n+1}\int_0^1 \frac{r^n}{t^{k-1}[k-(k+1)\,zt]}\,dr, \tag{6}$$

where now $t(0)=0$ and $t(1)=1$.

3. ASYMPTOTIC EXPANSIONS

We can approximate (1) by Stirling's series (cf. [5, p. 253]),

$$\Gamma(n+1)=e^{-n}n^n\sqrt{2n\pi}\left(1+\frac{1}{12n}+\frac{1}{288n^2}+O\left(\frac{1}{n^3}\right)\right), \qquad n\to\infty,$$

to get

$$\int_0^{1/z} [f(t)]^n\,dt=\frac{1}{z^{kn+1}}\left(\frac{k^k}{(k+1)^{k+1}}\right)^n \frac{\sqrt{2k\pi}}{(k+1)^{3/2}\sqrt{n}} \times\left\{1+\frac{k^2-5k+1}{12k(k+1)\,n}+O\left(\frac{1}{n^2}\right)\right\}. \tag{7}$$

For $k=1$ and $Re\{z\} > \frac{1}{2}$ we can obtain from (2) a full asymptotic expansion for $\int_{1/z}^1 [f(t)]^n\,dt$. Letting $w=4z(1-z)$, we use Newton's binomial series to get the formal expansion

$$\{1-wr\}^{-1/2}=\{(1-w)+w(1-r)\}^{-1/2} =\frac{1}{\sqrt{1-w}}\left\{1+\sum_{m=1}^{\infty}\binom{-\frac{1}{2}}{m}\frac{w^m}{(1-w)^m}(1-r)^m\right\},$$

where

$$\sqrt{1-w} = \sqrt{(1-2z)^2} = -(1-2z).$$

Inserting this expansion into (2) and integrating term by term, we arrive at the asymptotic series (cf. [3])

$$\begin{aligned}\int_{1/z}^{1} &[f(t)]^n \, dt \\ &\approx \frac{(1-z)^{n+1}}{1-2z} \left\{ \frac{1}{n+1} + \sum_{m=1}^{\infty} \binom{-\frac{1}{2}}{m} \frac{w^m}{(1-w)^m} B(n+1, m+1) \right\} \\ &= \frac{(1-z)^{n+1}}{1-2z} \left\{ \frac{1}{n+1} + \sum_{m=1}^{\infty} \binom{-\frac{1}{2}}{m} \frac{n!\, m!}{(n+m+1)!} \frac{w^m}{(1-w)^m} \right\}\end{aligned}$$

for $k=1$ and $Re\{z\} > \frac{1}{2}$.

For $k=1$ and $Re\{z\} < \frac{1}{2}$, we use (3) in a similar way to obtain the same asymptotic expansion

$$\begin{aligned}\int_{0}^{1} &[f(t)]^n \, dt \\ &\approx \frac{(1-z)^{n+1}}{1-2z} \left\{ \frac{1}{n+1} + \sum_{m=1}^{\infty} \binom{-\frac{1}{2}}{m} \frac{n!\, m!}{(n+m+1)!} \frac{w^m}{(1-w)^m} \right\},\end{aligned}$$

because now $\sqrt{1-w} = 1-2z$.

For general $k > 0$ we can use (5) and (6) to get asymptotic expansions for the integrals, but since t is determined only implicitly as a function of r, the results are less complete. Writing

$$t = t(r) = 1 + a_1(t-1) + a_2(r-1)^2 + \cdots$$

and referring to (4), we find that

$$t^k(1-zt) = (1-z) + [k-(k+1)\,z]\, a_1(r-1) + \cdots = (1-z)[1+(r-1)]$$

so that

$$a_1 = \frac{1-z}{k-(k+1)\,z}. \tag{8}$$

Expanding the factor

$$\frac{1}{t^{k-1}[k-(k+1)\,zt]}$$

into the power series about $r=1$, and using (8), we find after some calculations that

$$\int_{1/s}^{1} [f(t)]^n \, dt$$

$$= \frac{(1-z)^{n+1}}{k-(k+1)\,z} \left\{ \frac{1}{n+1} + \frac{k(1-z)[k-1-(k+1)\,z]}{(n+1)(n+2)[k-(k+1)\,z]^2} + \cdots \right\}$$

for $Re\{z\} > \frac{k}{k+1}$ and

$$\int_{0}^{1} [f(t)]^n \, dt = \frac{(1-z)^{n+1}}{k-(k+1)\,z} \left\{ \frac{1}{n+1} + \frac{k(1-z)[k-1-(k+1)\,z]}{(n+1)(n+2)[k-(k+1)\,z]^2} + \cdots \right\}$$

for $Re\{z\} < \frac{k}{k+1}$.

4. ZEROS OF HYPERGEOMETRIC POLYNOMIALS

The above asymptotic formulas give a heuristic guide to the behavior of zeros of our hypergeometric polynomials, but for more rigorous analysis it must be shown that the error terms are uniform in the parameter z. We will address the problem by basing the discussion directly on our integral representations, taken over paths of steepest descent. The following lemma will be useful.

LEMMA 2. *The zeros of* $F(-n, kn+l; kn+2; z)$ *are contained in the disk* $|z| < n+1$.

The proof is based on the following classical theorem (cf. Marden [5, p. 136]).

ENESTRÖM–KAKEYA THEOREM. *If* $0 < a_0 < a_1 < \cdots < a_n$, *then all zeros of the polynomial*

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

lie in the unit disk $|z| < 1$.

Proof of Lemma 2. The hypergeometric polynomial has the form

$$F_n(z) = F(-n, kn+1; kn+2; z) = c_0 + c_1 z + \cdots + c_n z^n,$$

where

$$c_m = (-1)^m \binom{n}{m} \frac{kn+1}{kn+m+1}.$$

It can be shown (by computing a derivative) that the absolute values of the ratios

$$\frac{c_m}{c_{m-1}} = -\frac{n-m+1}{m} \frac{kn+m}{kn+m+1}, \qquad m = 1, 2, ..., n,$$

decrease as m increases. Since

$$\left| \frac{c_n}{c_{n-1}} \right| = \frac{k+1}{(k+1)\, n+1} > \frac{1}{n+1},$$

it follows that

$$-\frac{(n+1)\, c_m}{c_{m-1}} > 1, \qquad m = 1, 2, ..., n.$$

This says that the coefficients of the polynomial

$$p(z) = F_n(-(n+1)\, z) = a_0 + a_1 z + \cdots + a_n z^n$$

are positive and increasing: $0 < a_0 < a_1 < \cdots < a_n$. Thus Lemma 2 follows from the Eneström–Kakeya theorem.

We shall also need the following elementary lemma.

LEMMA 3. *The polynomial* $F(-n, kn+1; kn+2; z)$ *has at least one zero outside the unit circle* $|z| = 1$.

Proof. As seen in the proof of Lemma 2, the polynomial has the form $c_0 + c_1 z + \cdots + c_n z^n$, where

$$c_0 = 1, \qquad c_n = (-1)^n \frac{kn+1}{(k+1)\, n+1}.$$

Thus the product of all zeros has modulus $|\frac{c_0}{c_n}| > 1$.

We are now ready to investigate the asymptotic behavior of the zeros. For $Re\{z\} < \frac{k}{k+1}$, we know from (6) that

$$F(-n, kn+1; kn+2; z) = (kn+1) \int_0^1 [f(t)]^n \, dt$$
$$= (kn+1)(1-z)^{n+1} \int_0^1 \frac{r^n}{t^{k-1}[k-(k+1)\,zt]} \, dr,$$

where $t = t(r)$ follows the path of steepest ascent of $|f|$ from 0 to 1, with $t(0) = 0$ and $t(1) = 1$.

This path is determined implicitly by (4). Substituting (4) into the integral, we see that any zeros $z = z_{nj}$ of $F(-n, kn+1; kn+2; z)$ in the half-plane $Re\{z\} < \frac{k}{k+1}$ must satisfy

$$n \int_0^1 \frac{(1-zt)\, tr^{n-1}}{k-(k+1)\, zt} \, dr = 0.$$

If the zeros are further restricted by the inequality $|z - \frac{k}{k+1}| \geqslant \varepsilon$ for some $\varepsilon > 0$, then the path of integration $t = t(r)$ is bounded away from the saddle point $\frac{k}{(k+1)\,z}$, and the denominator of the integrand satisfies $|k-(k+1)\,zt| \geqslant \delta > 0$, where δ is independent of z. Thus for any fixed ρ with $0 < \rho < 1$, we have by Lemma 2

$$n \left| \int_0^\rho \frac{(1-zt)\, tr^{n-1}}{k-(k+1)\, zt} \, dr \right| \leqslant Cn^2 \int_0^\rho r^{n-1} \, dr = Cn\rho^n \to 0 \tag{9}$$

as $n \to \infty$.

On the other hand, we will now show that for ρ sufficiently close to 1, the integral

$$n \int_\rho^1 \frac{(1-zt)\, tr^{n-1}}{k-(k+1)\, zt} \, dr \tag{10}$$

is bounded away from zero. We know that $Re\{zt\} < \frac{k}{k+1}$, since the path $t = t(r)$ must lie on the same side of the "continental divide" (cf. proof of Lemma 1) as the point 1. Our restriction on z further ensures that $|zt - \frac{k}{k+1}| > \frac{\varepsilon}{2}$ for t sufficiently near 1. But the linear fractional mapping

$$\omega = \phi(\zeta) = \frac{1-\zeta}{k-(k+1)\,\zeta}$$

sends the region

$$L_{\varepsilon/2}=\left\{\zeta : Re\{\zeta\} \leqslant \frac{k}{k+1},\ \left|\zeta-\frac{k}{k+1}\right| \geqslant \frac{\varepsilon}{2}\right\}$$

onto a semidisk to the right of the vertical line $Re\{\omega\}=\frac{1}{k+1}$. It follows that

$$Re\left\{\frac{(1-zt)\,t}{k-(k+1)\,zt}\right\}>\frac{1}{2(k+1)}$$

when t is close enough to 1. This shows that the real part of (10) is greater than

$$\frac{n}{2(k+1)}\int_{\rho}^{1} r^{n-1}\,dr>\frac{1}{4(k+1)}$$

for ρ near 1 and all z in the region L_{ε}. Combining this with (9), we see that for sufficiently large n the polynomial $F(-n, kn+1; kn+2; z)$ can have no zeros in L_{ε}. Thus any zeros in the half-plane $Re\{z\} \leqslant \frac{k}{k+1}$ must converge uniformly to the point $\frac{k}{k+1}$ as $n\to\infty$. In fact, numerical evidence suggests that the polynomial never has zeros in this half-plane, but we are unable to prove it. In view of Lemma 3, our asymptotic result shows at least that for n sufficiently large, the polynomial has at least one zero in the half-plane $Re\{z\}>\frac{k}{k+1}$. Later we will be able to conclude that many other zeros are in this half-plane when n is large.

We now turn to the case $Re\{z\}>\frac{k}{k+1}$. Our previous discussion shows that each zero $z=z_{nj}$ of $F(-n, kn+1; kn+2; z)$ in that half-plane will satisfy

$$\int_{0}^{1/z}[f(t)]^{n}\,dt+\int_{1/z}^{1}[f(t)]^{n}\,dt=0,$$

where the integrals are taken over paths of steepest ascent or descent. The first integral is approximated by the asymptotic series (7), while the second has the form (5) with $t=t(r)$ determined by (4) and $t(0)=\frac{1}{z}$, $t(1)=1$. Introducing (4) into (5) and invoking (7), we can write

$$\begin{aligned} z^{kn+1}(1-z)^{n}\int_{0}^{1}\frac{(1-z)\,tr^{n-1}}{k-(k+1)\,zt}\,dr \\ =-\left(\frac{k^{k}}{(k+1)^{k+1}}\right)^{n}\frac{\sqrt{2k\pi}}{(k+1)^{3/2}\sqrt{n}}\left\{1+O\left(\frac{1}{n}\right)\right\}. \end{aligned} \tag{11}$$

We shall suppose again that $|z-\frac{k}{k+1}| \geqslant \varepsilon > 0$. From (11) we may infer that the zeros do not cluster at the point 1. For if some sequence of zeros tends to 1, then by taking nth roots of moduli on both sides of (11) and letting $n \to \infty$, we can deduce that the left-hand side tends to zero while the right-hand side does not. Thus we may also assume that $|z-1| \geqslant \varepsilon$.

An argument similar to that already given for the case $Re\{z\} < \frac{k}{k+1}$ now shows that the integral

$$n \int_0^1 \frac{(1-zt)\, tr^{n-1}}{k-(k+1)\, zt}\, dr \tag{12}$$

remains bounded away from zero and infinity, uniformly in z, as $n \to \infty$. Thus by equating moduli in (11) and taking nth roots, we find as $n \to \infty$ that the zeros tend uniformly to the right-hand branch of the lemniscate

$$|z^k(1-z)| = \frac{k^k}{(k+1)^{k+1}}, \qquad Re\{z\} \geqslant \frac{k}{k+1}. \tag{13}$$

In particular, an argument similar to that given for $Re\{z\} < \frac{k}{k+1}$ now shows that the integral (12) has the form

$$n \int_\rho^1 \frac{(1-zt)\, tr^{n-1}}{k-(k+1)\, zt}\, dr + O(\rho^n), \qquad n \to \infty, \tag{14}$$

uniformly for zeros $z = z_{nj}$ satisfying

$$Re\{z\} > \frac{k}{k+1}, \qquad \left| z - \frac{k}{k+1} \right| \geqslant \varepsilon > 0. \tag{15}$$

But those zeros approach the lemniscate (13), while $t(r) \to 1$ as $r \to 1$. It therefore follows that for ρ near 1 and n sufficiently large, the factor

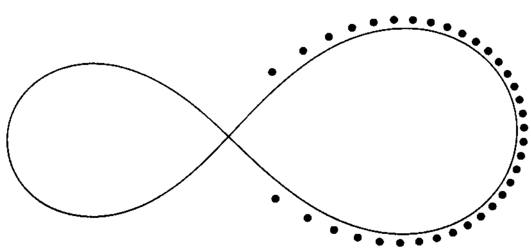

FIG. 2. Zeros of $F(-35, 29; 30; z)$ and lemniscate $|z^k(1-z)| = k^k/(k+1)^{k+1}$, $k = 0.8$.

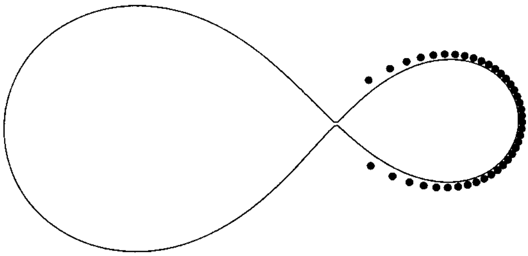

FIG. 3. Zeros of $F(-40, 69; 70; z)$ and lemniscate $|z^k(1-z)| = k^k/(k+1)^{k+1}$, $k = 1.7$.

$\frac{(1-zt)\,t}{k-(k+1)\,zt}$ in (14) is arbitrarily close to the image under the mapping $\omega = \phi(\zeta) = \frac{1-z\zeta}{k-(k+1)\,\zeta}$ of that portion of the half-lemniscate (13) for which $|z - \frac{k}{k+1}| \geqslant \varepsilon > 0$. This image is shaped like one branch of a hyperbola; it is a bounded curve with $|\arg w| < \frac{3\pi}{4}$. As $n \to \infty$, the integral (12) tends to this curve, and so with the choice of principal branch for the nth root,

$$\lim_{n \to \infty} \sqrt[n]{n} \left\{ \int_0^1 \frac{(1-zt)\,tr^{n-1}}{k-(k+1)\,zt}\, dr \right\}^{1/n} = 1,$$

uniformly for all zeros $z = z_{nj}$ in the region (15).

On the other hand, we have seen that for every n sufficiently large, the hypergeometric polynomial has at least one zero in the half-plane $Re\{z\} > \frac{k}{k+1}$. Such a zero must satisfy (11). But if we now take nth roots of both sides of (11), we see that (for large n) there are n points satisfying (11), distinguished by the n choices of $\sqrt[n]{-1}$. All of these points are zeros of the polynomial, spread out near the right-hand branch of the lemniscate. By suitable choice of $\sqrt[n]{-1}$, a sequence of values can be made to approach any given point $e^{i\theta}$ on the unit circle as $n \to \infty$. The corresponding zeros $z = z_{nj}$ will then approach the point on the right-hand branch of the lemniscate determined by $z^k(1-z) = e^{i\theta}(k^k/(k+1)^{k+1})$. Thus each point of the half-lemniscate is the limit of some sequence of zeros.

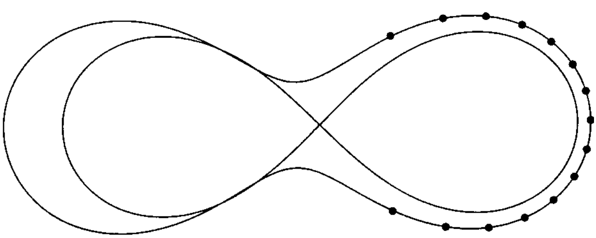

FIG. 4. Zeros of $F(-15, 16; 17; z)$, corresponding fitted lemniscate, and lemniscate $|z(1-z)| = \frac{1}{4}$.

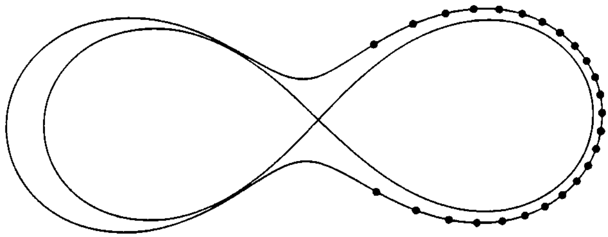

FIG. 5. Zeros of $F(-25, 26; 27; z)$, corresponding fitted lemniscate, and lemniscate $|z(1-z)| = \frac{1}{4}$.

Figures 2 and 3, generated by *Mathematica*, illustrate this result for $n = 35, 40$ and $k = 0.8, 1.7$.

5. FITTED LEMNISCATES

A close inspection of Figs. 2 and 3 suggests that all of the zeros of $F(-n, kn+1; kn+2; z)$ actually lie on other lemniscates of the form

$$|(z-a)^k (z-b)| = c,$$

where a, b, and c are real parameters. Figures 4, 5, and 6 show lemniscates of this form fitted to the zeros by requiring the curve to pass through the three points in the upper half-plane $Im\{z\} \geqslant 0$ farthest to the right, with parameters $k = 1$ and $n = 15, 25, 35$ respectively. All zeros appear to lie on these lemniscates, and the fitted lemniscates appear to approach the asymptotic lemniscate $|z(z-1)| = \frac{1}{4}$ as $n \to \infty$. Such regular behavior seems to us unlikely, but we have investigated the question numerically as well as graphically and have not been able to rule it out.

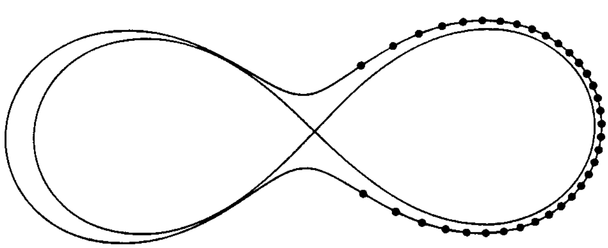

FIG. 6. Zeros of $F(-35, 36; 37; z)$, corresponding fitted lemniscate, and lemniscate $|z(1-z)| = \frac{1}{4}$.

REFERENCES

1. P. B. Borwein and W. Chen, Incomplete rational approximation in the complex plane, *Constr. Approx.* **11** (1995), 85–106.
2. N. G. de Bruijn, “Asymptotic Methods in Analysis,” North-Holland, Amsterdam, 1958.
3. E. T. Copson, “Asymptotic Expansions,” Cambridge University Press, London, 1965.
4. K. Driver and P. Duren, Asymptotic zero distribution of hypergeometric polynomials, *Numer. Algorithms* **21** (1999), 147–156.
5. M. Marden, “Geometry of Polynomials,” American Mathematical Society, Providence, RI, 1966.
6. E. T. Whittaker and G. N. Watson, “Modern Analysis,” Macmillan, New York, 1943.